1. [6 marks]

(a) Given the following matrix equation:

$$\begin{bmatrix} 2 & 4 \\ 3 & n \end{bmatrix}\begin{bmatrix} m \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -13 \end{bmatrix},$$

Determine the value of m and n.

[3]

$2m + 8 = -2$

$\underline{\underline{m = -5}}$

$3m + 2n = -13$

$-15 + 2n = -13$

$\underline{\underline{n = 1}}$

✓ equation involving m only.

✓ equation involving m and n.

✓ Correct solutions to both equations.

(b) If **P** and **Q** are square matrices and **PQ** = **P** + **Q**, then determine **P** in terms of **Q**.

[3]

$PQ - PI = Q$

$P(Q - I) = Q$

$\underline{\underline{P = Q(Q-I)^{-1}}}$

(Assuming $(Q-I)^{-1}$ exists)

✓ Rearranging equation appropriately.

✓ factoring out P using I.

✓ correct expression for P,

6

2. [4 marks]

Simplify $\left[\sqrt{3}\, cis\left(\frac{5\pi}{6}\right)\right]^3 \times \sqrt{3\, cis\left(\frac{\pi}{4}\right)}$

$= (\sqrt{3})^3 \; cis\left(\frac{15\pi}{6}\right) \times \sqrt{3} \; cis\left(\frac{\pi}{8}\right)$

$= 3\sqrt{3} \; cis \; \frac{5\pi}{2} \times \sqrt{3} \; cis\left(\frac{\pi}{8}\right)$

$= 9 \; cis\left(\frac{5\pi}{2} + \frac{\pi}{8}\right)$

$= 9 \; cis\left(\frac{21\pi}{8}\right)$

$= \underline{\underline{9 \; cis\left(\frac{5\pi}{8}\right)}}$

✓ correct expansion of $\left[\sqrt{3}\; cis\left(\frac{5\pi}{6}\right)\right]^3$.

✓ Correct expansion of $\sqrt{3\; cis\left(\frac{\pi}{4}\right)}$.

✓ Correct multiplication process on complex numbers.

✓ Answer provided in correct convention.

4

3. [7 marks]

For each of the following functions, find $\frac{dy}{dx}$.

(a) $y = x^3 \ln(\cos 2x)$ [in terms of x] [3]

$$\frac{dy}{dx} = 3x^2 \cdot \ln(\cos(2x)) + x^3 \left(\frac{-2\sin 2x}{\cos 2x}\right)$$

$$= x^2 \left[3 \ln(\cos(2x)) - 2x \tan(2x) \right]$$

✓ Correct use of product rule.

✓ Correct logarithmic differentiation.

✓ Appropriately simplified answer.

(b) $x = \frac{1-t}{1+t}$ and $y = 1+t$ [in terms of y] [4]

$$\frac{dx}{dt} = \frac{-(1+t) - (1-t)}{(1+t)^2}$$

$$= \frac{-2}{(1+t)^2}$$

$$\frac{dy}{dt} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{-2}{(1+t)^2}}$$

$$= \frac{-(1+t)^2}{2}$$

$$= -\frac{1}{2}y^2$$

✓ differentiates y w.r.t. t.

✓ differentiates x w.r.t. t.

✓ finds $\frac{dy}{dx}$ in terms of t

✓ writes $\frac{dy}{dx}$ in terms of y.

7

4. [6 marks]

(a) The locus of the complex number z satisfies the equation $|z+1| = |\bar{z}|$.
Determine the Cartesian equation of this locus. [4]

let $z = x + iy$

$\therefore |(x+iy)+1| = |x - iy|$

$\Rightarrow (x+1)^2 + y^2 = x^2 + y^2$

$x^2 + 2x + 1 = x^2$

$\therefore 2x + 1 = 0$

$x = -\frac{1}{2}$

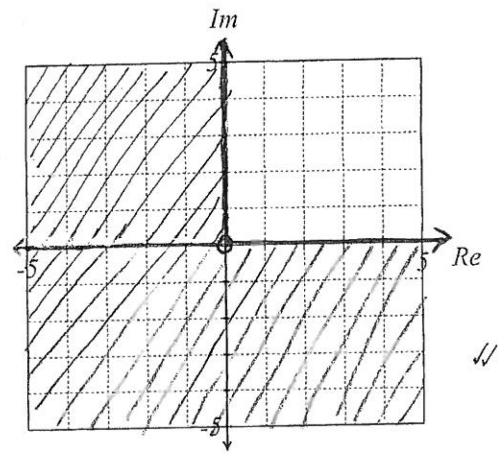
✓ re-writes z as $x+iy$.

✓ correctly expands LHS in terms of x and y.

✓ Correctly expands RHS in terms of x and y.

✓ Answer provided as a linear equation.

(b) Given that $0 \le Arg(z) \le 2\pi$, sketch, on the Argand plane below, the region defined by $\{z \mid \operatorname{Arg}(z^2) \ge \pi\}$. [2]



✓✓ correct region shaded.

6

5. [7 marks]

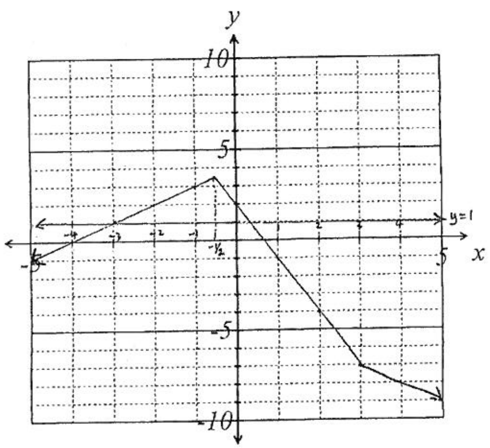
The function $f(x)$ is defined by $f(x) = |x-3| - |2x+1|$.

(a) Rewrite $f(x)$ in piecewise defined form. [3]

$$f(x) = \begin{cases} x+4 & , \quad x < -\frac{1}{2} \\ -3x+2 & , \quad -\frac{1}{2} \le x \le 3 \\ -x-4 & , \quad x > 3 \end{cases}$$

✓ correct "fracture" points.
✓ correct linear equations.
✓ correct domains listed.

(b) On the grid below, sketch the graph of $y = f(x)$. [2]



✓ correct fracture points shown.
✓ all three sections graphed correctly.

(c) For what values of x is $|x-3| - |2x+1| \ge 1$? [2]

$-3x+2 = 1$

$x = \frac{1}{3}$

$\therefore \quad \underline{\underline{-3 \le x \le \frac{1}{3}}}$

✓ Correct lower limit.
✓ Correct upper limit.

7

6. [8 marks]

Using the method of proof by exhaustion, prove that $x^5 - x$ is divisible by 5 for integers $x \ge 1$.

Let $x = 5n$, $5n \pm 1$ or $5n \pm 2$, where $n \in \mathbb{Z}$

If $x = 5n$:

$$x^5 - x = (5n)^5 - (5n)$$
$$= 5n\left((5n)^4 - 1\right) \quad \text{which is a multiple of 5.}$$

If $x = 5n \pm 1$:

Note → $x^5 - x = x(x^4 - 1) = x(x^2+1)(x^2-1)$

$$\therefore \quad x^5 - x = (5n\pm1)\left[(5n\pm1)^2+1\right]\left[(5n\pm1)^2-1\right]$$
$$= (5n\pm1)\left[(5n\pm1)^2+1\right]\left[25n^2 \pm 10n + 1 - 1\right]$$
$$= (5n\pm1)\left[(5n\pm1)^2+1\right]\left[25n^2 \pm 10n\right]$$
$$= 5(5n\pm1)\left[(5n\pm1)^2+1\right]\left[5n^2 \pm 2n\right]$$

which is a multiple of 5.

If $x = 5n \pm 2$:

$$x^5 - x = (5n\pm2)\left[(5n\pm2)^2+1\right]\left[(5n\pm2)^2-1\right]$$
$$= (5n\pm2)\left[25n^2 \pm 20n + 5\right]\left[(5n\pm2)^2-1\right]$$
$$= 5(5n\pm2)\left[5n^2 \pm 4n + 1\right]\left[(5n\pm2)^2-1\right]$$

which is a multiple of 5

$\therefore$ $\underline{x^5 - x \text{ is divisible by 5 for all integers } x \ge 1}$.

✓ re-writing x as $5n$, $5n\pm1$ and $5n\pm2$.
✓ Correct algebraic manipulation for $x = 5n$.
✓ factorises x^5-x as $x(x^2+1)(x^2-1)$.
✓ correct expansion for $x = 5n\pm1$
✓ correct conclusion for $x = 5n\pm1$
✓ correct expansion for $x = 5n\pm2$
✓ correct conclusion for $x = 5n\pm2$
✓ Final conclusive statement.

7. [4 marks]

Determine $\lim_{x \to \frac{\pi}{2}} \dfrac{2\cos x}{x - \frac{\pi}{2}}$ showing your full reasoning.

let $y = x - \frac{\pi}{2}$

$\therefore$ As $x \to \frac{\pi}{2}$, $y \to 0$

So, $\lim_{x \to \frac{\pi}{2}} \dfrac{2\cos x}{x - \pi/2}$

$= \lim_{y \to 0} \dfrac{2\cos\left(y + \frac{\pi}{2}\right)}{y}$

$= \lim_{y \to 0} \dfrac{-2\sin y}{y}$

$= \underline{\underline{-2}}$ $\quad \left(\text{since } \lim_{y \to 0} \dfrac{\sin y}{y} = 1\right)$

✓ Correct substitution for $x - \frac{\pi}{2}$.

✓ re-writes limit in terms of new, substituted pronumeral.

✓ converts $\cos\left(y + \frac{\pi}{2}\right)$ into $-\sin y$.

✓ correct limit as answer.

8. [8 Marks]

Given that $a = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$ and $c = \begin{pmatrix} 3 \\ m \\ 0 \end{pmatrix}$,

(a) determine the value of m such that b is perpendicular to c. [2]

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$

$-6 + 3m = 0$

$\therefore \underline{\underline{m = 2}}$

✓ uses dot product.

✓ correct solution for m.

(b) show that the cosine of the angle between a and c, in terms of m, is $\dfrac{m-6}{\sqrt{6m^2 + 54}}$. [3]

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{|\underset{\sim}{a}| \times |\underset{\sim}{c}|}$

$= \dfrac{-6 + m}{(\sqrt{6})(\sqrt{m^2 + 9})}$

$= \dfrac{m - 6}{\sqrt{6m^2 + 54}}$

✓ correct dot product.

✓ magnitudes of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ calculated correctly.

✓ Correct simplification.

(c) determine vector d such that the magnitude of d is 3 times the magnitude of b and in the same direction as a. [3]

$\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$3|\underset{\sim}{b}| = 3\sqrt{22}$

$\therefore \underset{\sim}{d} = \underline{\underline{\dfrac{3\sqrt{22}}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}}$ OR $\dfrac{\sqrt{132}}{2} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

✓ unit vector for $\underset{\sim}{a}$.

✓ magnitude of $\underset{\sim}{b}$.

✓ $\underset{\sim}{d}$ provided in correct form.

8

9. [8 marks]

From four numbers, three are chosen, averaged and the fourth one added. This can be done four ways, leaving out a different number each time. The four results are 17, 21, 23 and 29.
Let the four numbers be represented by a, b, c and d.

(a) Write down four equations involving the variables a, b, c and d. [4]

$$\left.\begin{aligned} \frac{a+b+c}{3} + d &= 17 \\ \frac{b+c+d}{3} + a &= 21 \\ \frac{a+c+d}{3} + b &= 23 \\ \frac{a+b+d}{3} + c &= 29 \end{aligned}\right\} \Rightarrow \begin{aligned} a+b+c+3d &= 51 \\ 3a+b+c+d &= 63 \\ a+3b+c+d &= 69 \\ a+b+3c+d &= 87 \end{aligned}$$

✓✓✓✓ writes 4 separate equations involving a, b, c and d.

(b) Use an inverse matrix method to determine the value of the four numbers. Show clearly all the matrices involved. [4]

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 51 \\ 63 \\ 69 \\ 87 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 51 \\ 63 \\ 69 \\ 87 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 12 \\ 21 \\ 3 \end{bmatrix}$$

✓ sets up appropriate matrix equation.
✓ pre-multiplies by the inverse of the coefficient matrix.
✓ calculates answer as a matrix.
✓ contextualises the answer.

$\therefore$ The four numbers are 3, 9, 12 and 21.

8

10. [9 marks]

Consider the three vectors:

$a = i - j + 2k$, $b = i + 2j + mk$, and $c = i + j - k$, where m is real.

(a) Determine the exact value(s) of m for which $|b| = 2\sqrt{3}$. [2]

$$1^2 + 2^2 + m^2 = 12$$

$$\therefore m^2 = 7$$

$$\Rightarrow m = \pm\sqrt{7}$$

✓ equates magnitude of $\underset{\sim}{b}$ and $2\sqrt{3}$.
✓ solves for m, giving both sol^ns.

(b) Find the value of m (to two decimal places) such that the acute angle between a and b is 45°. [2]

$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} = \frac{1}{\sqrt{2}}$$

$$\text{i.e., } \frac{2m-1}{(\sqrt{6})(\sqrt{m^2+5})} = \frac{1}{\sqrt{2}}$$

$$\therefore m = 6.24 \quad (2\text{ d.p.})$$

✓ equates $\frac{a\cdot b}{|a||b|}$ and cos 45°
✓ solves for m.

(c) A is the point defined by the position vector a, B is the point defined by the position vector b and C is the point defined by the position vector c.

(i) Determine the vector equation, in terms of m, of the line which passes through the points A and B. [3]

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ m \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ m-2 \end{pmatrix}$$

$\therefore$ Line passing through A and B:

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ m-2 \end{pmatrix}$$

✓ calculates $\overrightarrow{AB}$.
✓ uses appropriate point on line.
✓ provides answer in appropriate format.

7

(ii) Determine the vector equation of the plane, in terms of m, which contains all three points A, B and C.

[2]

$$\vec{AC} = \begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\2\\-3\end{pmatrix}$$

$\therefore$ Eqn of plane : $\underset{\sim}{r} = \begin{pmatrix}1\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\m-2\end{pmatrix} + \mu\begin{pmatrix}0\\2\\-3\end{pmatrix}$

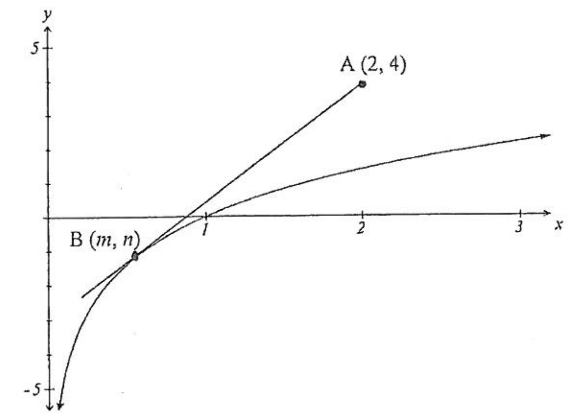
✓ calculates $\vec{AC}$

✓ provides eqn. of plane in appropriate format.

2

11. [9 marks]

The diagram below shows the graph of $y = 2\ln x$, the point A (2, 4) and the tangent from A meeting the curve at B with coordinates (m, n).



(a) Determine an expression for the gradient of the curve at point B, in terms of m.

[2]

$$\frac{dy}{dx} = \frac{2}{x}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=m} = \boxed{\frac{2}{m}}$$

✓ calculates $\frac{dy}{dx}$.

✓ calculates $\frac{dy}{dx}$ at (m, n).

(b) Determine the gradient of the line joining A to B, in terms of m.

[1]

$$\text{gradient} = \frac{4-n}{2-m} = \boxed{\frac{4-2\ln m}{2-m}}$$

✓ uses rise over run to determine gradient.

(c) Show that m satisfies the equation $6m - 2m\ln(m) - 4 = 0$.

[2]

$$\frac{4-2\ln m}{2-m} = \frac{2}{m}$$

$$\therefore 4m - 2m\ln m = 4 - 2m$$

$$\Rightarrow 6m - 2m\ln m - 4 = 0$$

✓ equates 2 expressions from (a) and (b).

✓ simplifies correctly.

5

(d) Hence, or otherwise, determine the equation of the tangent from **A** to the curve.

[4]

$6m - 2m\ln m - 4 = 0$

$\Rightarrow m = 0.5582$ or 17.97 (CAS)

If $m = 0.5582$,

then eqn. of tangent : $y - 4 = \dfrac{2}{0.5582}(x-2)$

$\therefore$ $\underline{\underline{y = 3.5829x - 3.1659}}$

If $m = 17.97$,

then eqn. of tangent : $y - 4 = \dfrac{2}{17.97}(x-2)$

$\therefore$ $\underline{\underline{y = 0.1113x + 3.7774}}$

✓ Solves equation using CAS.
✓ Considers both solutions.
✓ calculates eqn of tangent for $m = 0.5582$.
✓ Calculates eqn of tangent for $m = 17.97$.

4

12. **[12 marks]**

The points **A** (-1, 0), **B** (2, 0) and **C** (-1, 3) form the vertices of a triangle.

(a) The three points A, B and C are transformed using the matrix $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ to produce the points A', B' and C'.

Describe the effect of the transformation matrix M.

[1]

Shear parallel to x-axis, shear factor 2.

✓ correct description.

(b) The points A', B' and C' undergo a shear parallel to the vertical axis with a factor of 2 to produce the points A'', B'' and C''.

As a result, the point C' is vertically translated by k units to produce the point C''. What is the value of k?

[4]

$M \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

let $N = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

So, $N \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ -2 & 4 & 13 \end{bmatrix}$

So C' is $(5, 3)$ and C'' is $(5, 13)$

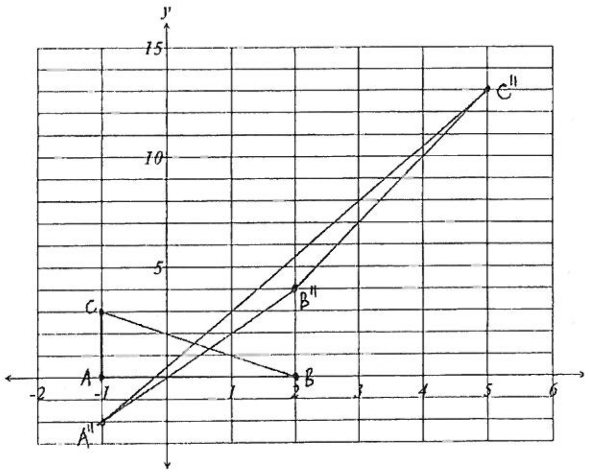
$\therefore$ C' has been translated 10 units up.

i.e. $\underline{\underline{k = 10}}$

✓ determines coordinates of C'.
✓ Uses correct matrix for vertical shear.
✓ determines coordinates of C''.
✓ calculates vertical translation.

5

(c) On the set of axes below, draw triangles ABC and A"B"C". [2]



✓ clearly labelled diagram of ΔABC.

✓ clearly labelled diagram of $\Delta A''B''C''$.

(d) What is the ratio of the area of triangle A''B''C'' to the area of triangle ABC? [2]

$\det(M) = 1$ and $\det(N) = 1$

$$\therefore \frac{\text{Area } \Delta A''B''C''}{\text{Area } \Delta ABC} = \underline{\underline{1}}$$

✓ calculates det(M) and det(N).

✓ determines ratio of areas.

(e) What single transformation matrix will map A'' back to A? [3]

$$(NM)^{-1} = \left(\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\right)^{-1}$$

$$= \begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}^{-1}$$

$$= \underline{\underline{\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}}}$$

✓ determines matrix NM.

✓ uses inverse for backward mapping.

✓ correct matrix provided.

7

13. [5 marks]

If a curve is defined by the rule $y = \sqrt{\dfrac{2x+1}{2x^2-1}}$, use logarithmic differentiation to determine the exact equation of the tangent to the curve at the point $(1, \sqrt{3})$.

$$\ln y = \frac{1}{2}\left[\ln(2x+1) - \ln(2x^2-1)\right]$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{2x+1} - \frac{4x}{2x^2-1}\right]$$

When $x=1$ and $y=\sqrt{3}$,

$$\frac{1}{\sqrt{3}}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{3} - 4\right]$$

$$\therefore \frac{dy}{dx} = \frac{-5}{3} \times \sqrt{3}$$

$$= \frac{-5\sqrt{3}}{3}$$

$$\therefore y - \sqrt{3} = \frac{-5\sqrt{3}}{3}(x-1)$$

$$\therefore y = \frac{-5\sqrt{3}}{3}x + \frac{5\sqrt{3}}{3} + \frac{3\sqrt{3}}{3}$$

$$\underline{\underline{y = \frac{-5\sqrt{3}}{3}x + \frac{8\sqrt{3}}{3}}}$$

OR $5\sqrt{3} + 3y - 8\sqrt{3} = 0$

✓ natural log. of both sides.

✓✓ implicit differentiation.

✓ calculates $\frac{dy}{dx}$ at $(1, \sqrt{3})$

✓ calculates eqn. of tangent.

5

14. [4 marks]

If the argument of the complex number w is $\frac{\pi}{3}$ and the argument of the complex number z is θ,

(a) Show that $\sin\left(\frac{\pi}{3} - \theta\right) = 0$. [2]

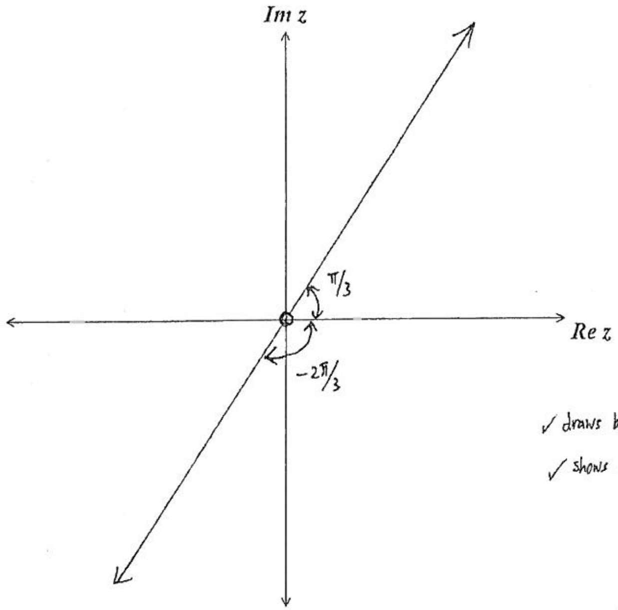
$$Arg\left(\frac{w}{z}\right) = \frac{\pi}{3} - \theta$$

since $\left(\frac{w}{z}\right)$ is Real, $Arg\left(\frac{w}{z}\right) = 0$

$$\therefore \quad \frac{\pi}{3} - \theta = 0$$

✓ calculates $Arg\left(\frac{w}{z}\right)$.
✓ shows why $Arg\left(\frac{w}{z}\right) = 0$.

(b) sketch the set of all complex numbers z for which $\frac{w}{z}$ is real. [2]



✓ draws both $\theta = \frac{\pi}{3}$ and $\theta = \frac{-2\pi}{3}$.
✓ shows angles clearly labelled.

4

15. [13 marks]

Two radio-controlled model planes take off at the same time from two different positions and with constant velocities. Model A leaves from the point with position vector $(-3i - 7j)$ metres and has velocity $(5i - j + 2k)$ m/s; model B leaves from the point with position vector $(7i - j - 8k)$ metres and has velocity $(3i - 4j + 6k)$ m/s.

(a) Determine the distance between the two planes after 2 seconds of flight. [3]

$$\underset{\sim}{r}_A(2) = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 4 \end{pmatrix}$$

$$\underset{\sim}{r}_B(2) = \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 13 \\ -9 \\ 4 \end{pmatrix}$$

$$\therefore \left|\overrightarrow{AB}\right|_{t=2} = \sqrt{(-6)^2 + 0^2 + 0^2}$$

$$= \underline{\underline{6\ m}}$$

✓ calculates $\underset{\sim}{r}_A(2)$.
✓ calculates $\underset{\sim}{r}_B(2)$.
✓ calculates distance.

(b) Show that the two planes do not collide with each other. [3]

$${}_A\underset{\sim}{r}_B(0) = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix}$$

$${}_B\underset{\sim}{V}_A = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$$

since $\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ is clearly not parallel to $\begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix}$

as $\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} \neq k\begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix}$ for unique k,

$\therefore$ A and B do not collide.

✓ Calculates $\overrightarrow{AB}$ and ${}_B V_A$
✓ shows that $\overrightarrow{AB} \not\parallel {}_B V_A$
✓ concludes no collision.

6

(c) Determine the shortest distance between the two model planes and the time at which this occurs.

[5]

$$ {}_A\underset{\sim}{r}_B(t) = \begin{pmatrix} -3+5t \\ -7-t \\ 2t \end{pmatrix} - \begin{pmatrix} 7+3t \\ -1-4t \\ -8+6t \end{pmatrix} $$

$$ = \begin{pmatrix} -10+2t \\ -6+3t \\ 8-4t \end{pmatrix} $$

✓ calculates ${}_A\underset{\sim}{r}_B(t)$.
✓ writes expression for $|{}_A\underset{\sim}{r}_B(t)|$.
✓ Use CAS to minimize $|{}_A\underset{\sim}{r}_B(t)|$.
✓ determines min. distance.
✓ determines time of min. dist.

$$ |{}_A\underset{\sim}{r}_B(t)| = \sqrt{(-10+2t)^2 + (-6+3t)^2 + (8-4t)^2} $$

use CAS: min value of $|{}_A\underset{\sim}{r}_B(t)| = \underline{\underline{5.57}}$ m

and occurs when $\underline{\underline{t = 2.41 \text{ seconds}}}$

(d) If the velocity of model B is $(qi - 4j + 6k)$ m/s, determine the value of q such that the two planes do in fact collide.

[2]

$$ {}_B\underset{\sim}{V}_A = \begin{pmatrix} q-5 \\ -3 \\ 4 \end{pmatrix} $$

For collision, ${}_B\underset{\sim}{V}_A = k \cdot {}_A\underset{\sim}{r}_B(0)$

$$ \begin{pmatrix} q-5 \\ -3 \\ 4 \end{pmatrix} = k\begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix} $$

$k = 0.5$

$\therefore \quad q - 5 = -5$

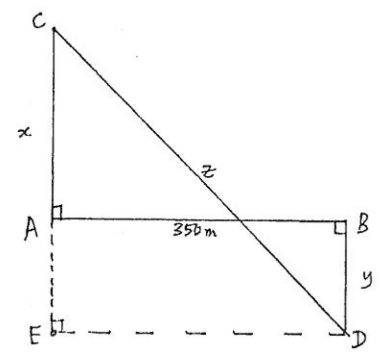
$\underline{\underline{q = 0}}$

✓ Appropriate test for collision with new ${}_B\underset{\sim}{V}_A$.
✓ Solves for q.

7

16. [8 marks]

Person A and Person B are both on bikes which are separated by 350 meters. B is due east of A. Person A starts riding north at a rate of 5 m/s and 7 minutes later Person B starts riding south at 3 m/s.
At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?



$$ CD^2 = CE^2 + ED^2 $$

$$ z^2 = (x+y)^2 + 350^2 = x^2 + 2xy + y^2 + 350^2 $$

$$ 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2\left(y\cdot\frac{dx}{dt} + x\cdot\frac{dy}{dt}\right) + 2y\cdot\frac{dy}{dt} $$

Now: when $t = 25 \times 60$, $x = 7{,}500$ m

when $t = 18 \times 60$, $y = 3{,}240$ m

$\therefore z = 10{,}745.70$ m (2 dp)

$$ \therefore \frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) $$

$$ = \frac{7500 + 3240}{10745.70}(5+3) $$

$$ = 7.9957 $$

$$ \approx \underline{\underline{8.00 \text{ m/s}}} \quad (2 \text{ dp}) $$

✓ Clearly labelled diagram.
✓✓ determines equation involving z, x and y.
✓✓ implicit differentiation
✓ calculates x, y and z at appropriate time.
✓✓ substitutes to determine $\frac{dz}{dt}$.

8

17. [5 marks]

Consider the equation $(|x| + a)^2 = 9$.

Showing full reasoning, determine the solution(s) to the equation, in terms of a.
State the conditions necessary for the existence of each of the solutions provided.

$|x| + a = \pm 3$

Case 1: $|x| + a = 3$, $x > 0$

$\therefore x = 3 - a$

✓ Considers cases for $x>0$ and $x<0$.

✓ considers cases where $|x|+a=3$.

case 2: $|x| + a = 3$, $x < 0$

✓ considers cases where $|x|+a=-3$.

$\therefore x = a - 3$

✓✓ correct solution with appropriate conditions.

case 3: $|x| + a = -3$, $x > 0$

$\therefore x = -3 - a$

case 4: $|x| + a = -3$, $x < 0$

$\therefore x = a + 3$

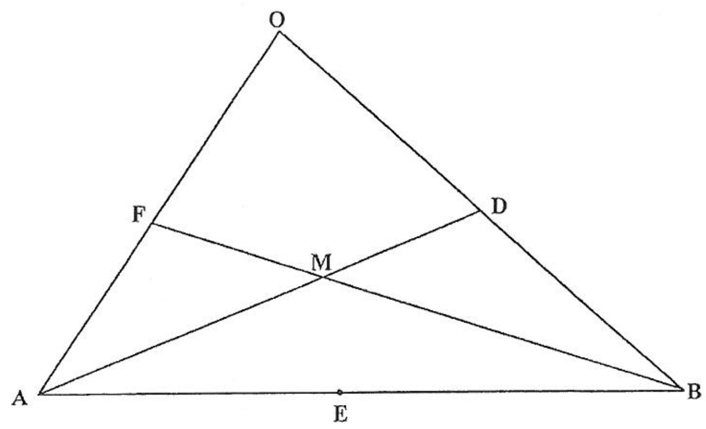
$\therefore$ $x = -a \pm 3$ for $x > 0$

and $x = a \pm 3$ for $x < 0$

5.

18. [12 marks]

OAB is a triangle with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.
Points D, E, and F are the midpoints of each side of the triangle.
$\overrightarrow{AM} = h\overrightarrow{AD}$ and $\overrightarrow{MF} = k\overrightarrow{BF}$



(a) Determine $\overrightarrow{AD}$ and $\overrightarrow{BF}$ in terms of a and b. [2]

$\overrightarrow{AD} = -\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\overrightarrow{BF} = -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

✓ determines $\overrightarrow{AD}$.
✓ determines $\overrightarrow{BF}$.

(b) Determine $\overrightarrow{AM}$ and $\overrightarrow{MF}$ in terms of a, b, h and k. [2]

$\therefore \overrightarrow{AM} = -h\underset{\sim}{a} + \frac{h}{2}\underset{\sim}{b}$

$\overrightarrow{MF} = -k\underset{\sim}{b} + \frac{k}{2}\underset{\sim}{a}$

✓ determines $\overrightarrow{AM}$.
✓ determines $\overrightarrow{MF}$.

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(c) Hence determine the value of h and of k. [5]

$$\vec{AM} + \vec{MF} = \vec{AF}$$

$$-h\underset{\sim}{a} + \tfrac{h}{2}\underset{\sim}{b} + \tfrac{k}{2}\underset{\sim}{a} - k\underset{\sim}{b} = -\tfrac{1}{2}\underset{\sim}{a}$$

comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

$$-h + \tfrac{k}{2} = -\tfrac{1}{2} \quad \text{and} \quad \tfrac{h}{2} - k = 0$$

solving gives $\underline{\underline{h = \tfrac{2}{3}}}$ and $\underline{\underline{k = \tfrac{1}{3}}}$

✓ uses appropriate relationship between $\vec{AM}$, $\vec{MF}$ and $\vec{AF}$.
✓ substitutes appropriately into this relationship.
✓ compares coefficients.
✓ determines h.
✓ determines k.

(d) Show that $\overline{OM} = \frac{2}{3}\overline{OE}$. [3]

$$\vec{OE} = \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

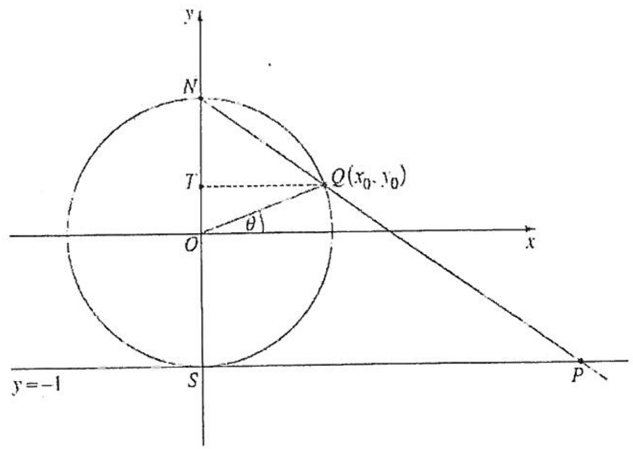
$$\vec{OM} = \vec{OA} + \vec{AM}$$
$$= \underset{\sim}{a} + \tfrac{2}{3}\left(-\underset{\sim}{a} + \tfrac{1}{2}\underset{\sim}{b}\right)$$
$$= \tfrac{1}{3}\underset{\sim}{a} + \tfrac{1}{3}\underset{\sim}{b}$$
$$= \tfrac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b})$$
$$= \tfrac{2}{3}\left[\tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})\right]$$
$$= \underline{\tfrac{2}{3}\vec{OE}}$$

✓ calculates $\vec{OE}$.
✓ calculates $\vec{OM}$.
✓ manipulates $\vec{OM}$ to show $\vec{OM} = \frac{2}{3}\vec{OE}$.

8

19. [9 marks]

In the diagram below, $Q(x_0, y_0)$ is a point on the unit circle $x^2 + y^2 = 1$ at an angle θ from the positive x-axis, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. The line through $N(0, 1)$ and Q intersects the line $y = -1$ at P. The points $T(0, y_0)$ and $S(0, -1)$ are on the y-axis.



(a) Show that $SP = \dfrac{2\cos\theta}{1 - \sin\theta}$. [2]

since ΔNTQ is similar to ΔNSP

$$\frac{NT}{NS} = \frac{TQ}{SP}$$

$$\therefore \frac{1 - \sin\theta}{2} = \frac{\cos\theta}{SP} \Rightarrow SP = \frac{2\cos\theta}{1 - \sin\theta} \; *$$

✓ uses notion of similarity.
✓ shows $SP = \frac{2\cos\theta}{1-\sin\theta}$.

(b) Show that $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1}{\cos\theta} + \tan\theta$. [2]

LHS: $$\frac{\cos\theta}{1 - \sin\theta} \times \frac{1 + \sin\theta}{1 + \sin\theta} = \frac{\cos\theta(1 + \sin\theta)}{1 - \sin^2\theta}$$
$$= \frac{\cos\theta(1 + \sin\theta)}{\cos^2\theta}$$
$$= \frac{1 + \sin\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta} + \tan\theta \; *$$

✓ rationalises denominator on LHS.
✓ concludes appropriately.

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(c) Show that $\angle SNP = \dfrac{\theta}{2} + \dfrac{\pi}{4}$.

[2]

$\angle SNP = \angle OQN \qquad (\Delta ONQ \text{ is isosceles.})$

$\therefore \angle SNP = \dfrac{\pi - \left(\frac{\pi}{2} - \theta\right)}{2}$

$= \dfrac{\frac{\pi}{2} + \theta}{2}$

$= \dfrac{\theta}{2} + \dfrac{\pi}{4}$ #

✓ uses fact that ΔONQ is isosceles.

✓ concludes appropriately.

(d) Hence, or otherwise, show that $\dfrac{1}{\cos\theta} + \tan\theta = \tan\left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right)$.

[3]

$\tan \angle SNP = \dfrac{SP}{2}$

$\therefore SP = 2\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$

From (a): $SP = \dfrac{2\cos\theta}{1-\sin\theta}$

$\therefore \dfrac{2\cos\theta}{1-\sin\theta} = 2\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$

from (b): $2\left(\dfrac{1}{\cos\theta} + \tan\theta\right) = 2\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$

$\therefore \dfrac{1}{\cos\theta} + \tan\theta = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$ #

✓ determines that $SP = 2\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

✓ equates with answer from (a).

✓ equates with answer from (b) to conclude appropriately.

5

20. [6 marks]

The line L has equation $r = \begin{pmatrix} -4+\lambda \\ -2\lambda \\ \lambda - 2 \end{pmatrix}$. The point A has position vector $\overrightarrow{OA} = \begin{pmatrix} k \\ -2 \\ 0 \end{pmatrix}$,

where $k > 0$.

If the shortest distance between the line L and the point A is $2\sqrt{5}$ units, determine the value of k. Show full reasoning.

Point B on the line L has pos. vector $\overrightarrow{OB} = \begin{pmatrix} -4+\lambda \\ -2\lambda \\ \lambda-2 \end{pmatrix}$

$\therefore \overrightarrow{AB} = \begin{pmatrix} -4-k+\lambda \\ 2-2\lambda \\ -2+\lambda \end{pmatrix}$

For $\overrightarrow{AB}$ to be $\perp$ to line L:

$\begin{pmatrix} -4-k+\lambda \\ 2-2\lambda \\ -2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$

$\Rightarrow -4-k+\lambda-4+4\lambda+\lambda-2 = 0$

$\therefore \lambda = \dfrac{k+10}{6}$

$\therefore |\overrightarrow{AB}| = \left|\begin{pmatrix} -4-k-\frac{k+10}{6} \\ 2-\frac{k+10}{3} \\ -2+\frac{k+10}{6} \end{pmatrix}\right| = 2\sqrt{5}$

CAS SOLVE: $k = 2$ or $k = -7.6$ (reject since $k>0$)

$\therefore \underline{\underline{k=2}}$

✓ determines $\overrightarrow{AB}$ or equivalent.

✓ uses dot product to set up an equation involving k and λ.

✓ solves λ in terms of k.

✓ substitutes λ into magnitude of $\overrightarrow{AB}$.

✓ solves equation using CAS.

✓ correct final answer.

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